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TORSION INJECTIVE COVERS AND RESOLVENTS

By

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1. Introduction

E. Enochs began the study of injective covers in [3], characterizing when any left R-module has an injective cover. This happens if and only if R is a left noetherian ring. Torsion injective covers and torsionfree injective covers were introduced by Ahsan and Enochs in [2] and [1] respectively, in the context of the Goldie torsion theory. B. Torrecillas in [13] defined τ-torsionfree τ-injective covers and τ-injective covers for τ a hereditary torsion theory. These covers have been studied in [6] and [7].

In this paper, τ-torsion τ-injective covers and envelopes are studied for τ any torsion theory. Then we construct relative homological algebra by means of complexes with this kind of covers and envelopes. In Section 3, we find necessary and sufficient conditions for the existence of τ-torsion τ-injective covers and envelopes for any module. In the aim of the descomposition theorem of abelian groups in divisible and reduced part, we give a torsion theoretical version in terms of τ-torsion τ-injective modules as divisibles ones, Proposition 2, relating such descomposition with certain condition on the existence of τ-torsion τ-injective covers. The existence of τ-torsion τ-injective envelopes is given in Theorem 2.

In Section 4, we study the balance (see [5]) of the functor $Hom(-,-)$ relative to the class of τ-torsion τ-injective modules. When the balance is given, it is possible to introduce left derived functor of $Hom(-,-)$ by using resolvents and resolutions of τ-torsion τ-injective modules. Left and right relative global dimension of the ring R are defined and analysed.

2. Preliminaries

Throught this note R denotes a unitary ring, R-Mod the category of all left R-modules and all R-homomorphisms, and $\mathscr{C}$ a full subcategory of R-Mod

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closed under isomorphisms and direct summands. τ will denote a hereditary torsion theory on R-Mod and $\mathscr{L}(\tau)$ the Gabriel filter associated to τ. We mean by $(\mathscr{T}_\tau, \mathscr{F}_\tau)$ the τ-torsion and τ-torsionfree classes respectively associated to τ. By $Q_\tau(-)$, we denote the localization functor associated to τ and $\tau(-)$ the τ-torsion functor. We say that τ is *stable* if the class of τ-torsion left R-modules is closed under injective hulls.

We say that an R-module M is τ-injective if $Ext^1_R(T, M) = 0$ for all τ-torsion R-module T. We will denote by $E_\tau(M)$ the τ-injective hull of any left R-module M.

We will say that τ is a TTF-theory in case that the class of τ-torsion modules is closed under direct products. It is well known that τ is a TTF-theory if and only if there exists an idempotent two-sided ideal I such that $\mathscr{T}_\tau$ consists of those left R-modules M with $IM = 0$. Since $\mathscr{T}_\tau$ is a torsionfree class, we will denote by $(\mathscr{C}_\tau, \mathscr{T}_\tau, \mathscr{F}_\tau)$ the triple with $\mathscr{C}_\tau$ the torsion class associated to $\mathscr{T}_\tau$.

All R-modules will be left R-modules and all torsion theories will be hereditary. We will denote by $\mathscr{I}\mathscr{T}_\tau$ (respectively $\mathscr{I}_\tau$) the class of all τ-torsion τ-injective R-modules (resp. the class of all τ-injective R-modules). (For concepts about torsion theory we will refer to [8] and [10]).

We recall the definition introduced by E. Enochs in [3].

Definition 1. *Let M be an R-module. We say that E in $\mathscr{C}$ is a $\mathscr{C}$-precover of M if there exists an homomorphism $\phi: E \to M$ such that the triangle*

$$\begin{tikzcd} E' \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r, "\phi"] & M \end{tikzcd}$$

can be completed for each homomorphism $E' \to M$ with E' in $\mathscr{C}$. If the triangle

$$\begin{tikzcd} E \arrow[d, dotted] \arrow[dr, "\phi"] & \\ E \arrow[r, "\phi"] & M \end{tikzcd}$$

can be completed only by automorphisms, we say that $\phi: E \to M$ is a $\mathscr{C}$-cover.

REMARK. a) A $\mathscr{C}$-cover of an object, if it exists, is unique up to isomorphisms.

b) Dually, the concept of $\mathscr{C}$-envelope can be defined, (cf. [3]).

Now, we give the concepts of resolvent and resolution that appear in [3] for $\mathscr{C}$ the class of injectives modules.

DEFINITION 2. (a) *A complex* $\cdots \to E_1 \to E_0 \to M \to 0$ *where*

$$E_0 \to M, \quad E_1 \to Ker(E_0 \to M), \quad E_{n+1} \to Ker(E_n \to E_{n-1})$$

for $n \geq 1$ *are* $\mathscr{C}$*-precovers is called a* $\mathscr{C}$*-resolvent of* M*. If* $E_{n+1} \to Ker(E_n \to E_{n-1})$ *are* $\mathscr{C}$*-covers, the above complex is called a minimal* $\mathscr{C}$*-resolvent of* M*. For* $n \geq 0$, $C_n = Ker(E_n \to E_{n-1})$ *is called the nth syzygy of* M *(for* $n = 0$*, we take* $E_{n-1} = M$*).*

(b) *Dually,* $\mathscr{C}$*-resolutions are defined. If* $0 \to M \to G^0 \to G^1 \to \cdots$ *is a* $\mathscr{C}$*-resolution of* M*, then* $C^0 = M$*,* $C^n = Im(G^{n-1} \to G^n)$ *for* $n \geq 1$ *is called the nth* $\mathscr{C}$*-cosyzygy of* M*.*

We will be interested in $\mathscr{I}\mathscr{T}_\tau$-covers, $\mathscr{I}\mathscr{T}_\tau$-resolvents and $\mathscr{I}\mathscr{T}_\tau$-resolutions.

3. Existence of covers and envelopes

First, we study when there exist $\mathscr{I}\mathscr{T}_\tau$-covers.

LEMMA 1. *Let* τ *be a torsion theory in R-Mod. Suppose that* $\mathscr{L}(\tau)$ *verifies the ascending chain condition over left ideals. Then* $\mathscr{I}\mathscr{T}_\tau$ *is closed under direct limits and every R-module in* $\mathscr{I}\mathscr{T}_\tau$ *is a direct sum of indescomposable R-modules.*

PROOF. It is well-known that the full subcategory $\mathscr{T}_\tau$ is a Grothendieck category with the following set of generators: $\{R/I \mid I \in \mathscr{L}(\tau)\}$. Since $\mathscr{L}(\tau)$ verifies the ascending chain condition over ideals, it follows that for each $I \in \mathscr{L}(\tau)$, R/I is a noetherian object in $\mathscr{T}_\tau$. So $\mathscr{T}_\tau$ is a locally noetherian Grothendieck category. Hence, the class of injectives objects in $\mathscr{T}_\tau$ is closed under direct limits and every injective object in $\mathscr{T}_\tau$ is a direct sum of indecomposable objects (see [10, Page 124]). But, it is not hard to check that the injective objects in $\mathscr{T}_\tau$ are precisely the τ-torsion τ-injective R-modules. $\square$

The following Theorem extends the result of Ahsan and Enochs [2, Page 259].

THEOREM 1. *Let τ a torsion theory in R-Mod. The following conditions are equivalent.*

(a) *$\mathcal{L}(\tau)$ verifies the ascending chain condition over left ideals.*

(b) *Every R-module has a $\mathcal{I}\mathcal{T}_\tau$-cover.*

PROOF. (a)$\Rightarrow$(b) By [3], in order to find $\mathcal{I}\mathcal{T}_\tau$-covers for each R-module is enough to impose that every R-module in $\mathcal{I}\mathcal{T}_\tau$ can be written as a direct sum of indescomposable R-modules in $\mathcal{I}\mathcal{T}_\tau$, and $\mathcal{I}\mathcal{T}_\tau$ will be closed under direct limits. So, we only have to apply Lemma 1.

(b)$\Rightarrow$(a) Suppose that every R-module has a $\mathcal{I}\mathcal{T}_\tau$-cover. Then $\mathcal{I}\mathcal{T}_\tau$ is closed under direct sums [6, Proposition 1]. Hence, by [9, Lemma 2], $\mathcal{L}(\tau)$ verifies the ascending chain condition over ideals. □

REMARK. a) If M is a τ-torsionfree R-module, then $0 \to M$ is the $\mathcal{I}\mathcal{T}_\tau$-cover of M.

b) If $\tau(M)$ is τ-injective, then the inclusion $\tau(M) \to M$ is the $\mathcal{I}\mathcal{T}_\tau$-cover of M.

c) The $\mathcal{I}\mathcal{T}_\tau$-cover of M and $\tau(M)$ are the same, in case that it exists.

EXAMPLES. 1) Suppose that τ is a stable torsion theory. If $E \to M$ is an injective cover of M, then $\tau(E) \to M$ (the restriction map) is a $\mathcal{I}\mathcal{T}_\tau$-cover of M (see [6, Note 2]).

2) Let R be a commutative noetherian ring. It is well known that, in this case, every torsion theory is stable. In [4, Proposition 3.2] has been proved that every finitely generated R-module M has an injective cover in the form $E = \bigoplus_{i=1}^{n} E(R/\eta_i)$ where η_i is a maximal ideal for each i. Since, in this case, the $\mathcal{I}\mathcal{T}_\tau$-cover of M is the τ-torsion direct sumand of E, it follows that the $\mathcal{I}\mathcal{T}_\tau$-cover of M is $E(R/\eta_1) \oplus \cdots \oplus E(R/\eta_k)$, where $\eta_1, \dots, \eta_k$ are the maximal ideals in $\mathcal{L}(\tau)$ which appear in the decomposition of E.

3) Let (R, η) a commutative noetherian local ring with maximal ideal η and τ_η the torsion theory associated to the punctured spectrum of R, i.e. generated by the powers of η. For each finitely generated R-module M, its injective cover has the form $E(R/\eta)^{(s)}$. Since $E(R/\eta)^{(s)}$ is τ_η-torsion, it follows that the injective cover and the $\mathcal{I}\mathcal{T}_{\tau_\eta}$-cover coincide. (Note that this happens for any torsion theory).

In order to find $\mathcal{I}\mathcal{T}_\tau$-resolutions, we must answer when there exist $\mathcal{I}\mathcal{T}_\tau$-envelopes for any R-module. The next result gives a complete solution to this problem. Now, we give several useful lemmas.

LEMMA 2. *$\mathscr{I}_\tau$-envelopes and τ-injective hulls coincide for any module M (the same for $\mathscr{I}\mathscr{T}_\tau$-envelopes and τ-injective hulls of τ-torsion R-modules).*

PROOF. Easy. $\square$

The following Lemma can be proved by using the dual arguments of [3, Proposition 2.1].

LEMMA 3. *Let $\mathscr{C}$ be a class of R-modules closed under isomorphisms and direct summands. If every R-module has a $\mathscr{C}$-envelope, then $\mathscr{C}$ is closed under direct products.*

The following result characterizes the torsion theories with the property that any module has a τ-torsion τ-injective envelope.

THEOREM 2. *Let τ a torsion theory in R-Mod. The following conditions are equivalent.*

(a) *Every R-module has a $\mathscr{I}\mathscr{T}_\tau$-envelope.*
(b) *τ is a TTF-theory.*

PROOF. (a) $\rightarrow$ (b) Let $\{T_i\}_{i\in I}$ a index family of τ-torsion R-modules. By Lemma 3, $\prod_{i\in I} E_\tau(T_i)$ belongs to $\mathscr{I}\mathscr{T}_\tau$. Because $\prod_{i\in I} T_i$ is a submodule of $\prod_{i\in I} E_\tau(T_i)$, it follows that it is τ-torsion. Hence τ is a TTF-theory.

(b) $\rightarrow$ (a) Let $\tau \equiv (\mathscr{C}, \mathscr{T}, \mathscr{F})$ be a TTF-theory and σ the idempotent radical associated to the torsion theory (non necessarily hereditary) $(\mathscr{C}, \mathscr{T})$. We prove that for each R-module M, $M \xrightarrow{p} M/\sigma(M) \xrightarrow{i} E_\tau(M/\sigma(M))$ is the $\mathscr{I}\mathscr{T}_\tau$-envelope of M (where p is the natural projection and i is the inclusion). Also, we denote by $j: \sigma(M) \rightarrow M$ the inclusion. It is clear that $E_\tau(M/\sigma(M))$ is in $\mathscr{I}\mathscr{T}_\tau$. Let $X \in \mathscr{I}\mathscr{T}_\tau$ and let $g: M \rightarrow X$ be a morphism. Since $gj = 0$, there exists $\bar{g}: M/\sigma(M) \rightarrow X$ such that $\bar{g}p = g$. Since $M/\sigma(M) \xrightarrow{i} E_\tau(M/\sigma(M))$ is the $\mathscr{I}\mathscr{T}_\tau$-envelope of $M/\sigma(M)$, it follows that there exists $h: E_\tau(M/\sigma(M)) \rightarrow X$ such that $hi = \bar{g}$. Therefore $hip = g$ and $ip: M \rightarrow E_\tau(M/\sigma(M))$ is a $\mathscr{I}\mathscr{T}_\tau$-preenvelope of M. Finally, if $\alpha: E_\tau(M/\sigma(M)) \rightarrow E_\tau(M/\sigma(M))$ verifies $\alpha ip = p$, then, since p is epic, it follows that $\alpha i = i$. Because $i: M/\sigma(M) \rightarrow E_\tau(M/\sigma(M))$ is a $\mathscr{I}\mathscr{T}_\tau$-envelope, α is an automorphism. $\square$

Now, we are going to study when the $\mathscr{I}\mathscr{T}_\tau$-cover of any R-module is a monomorphism. Following [8, Chapter 11], an R-module L is said τ-projective if $Ext^1_R(L, T) = 0$ for every τ-torsion R-module T.

Proposition 1. *Let τ be a torsion theory in R-Mod. If $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals and every left ideal in $\mathscr{L}(\tau)$ is τ-projective, then any R-module M has a monic $\mathscr{I}\mathscr{T}_\tau$-cover.*

If τ is stable, the converse is also true.

Proof. By [6, Proposition 4], we only have to prove that the class $\mathscr{I}\mathscr{T}_\tau$ is closed under direct sums and epimorphic images. By Lemma 1 and since $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals, it follows that $\mathscr{I}\mathscr{T}_\tau$ is closed under direct sums.

Now, we are going to see that if every ideal in $\mathscr{L}(\tau)$ is τ-projective, then $\mathscr{I}\mathscr{T}_\tau$ is closed under epimorphic images. Let $0 \to K \to E \to E/K \to 0$ an exact sequence with E in $\mathscr{I}\mathscr{T}_\tau$. Given $I \in \mathscr{L}(\tau)$, we have the exact sequence:

$$\cdots \to Ext_R^1(R/I, E) \to Ext_R^1(R/I, E/K) \to Ext_R^2(R/I, K) \to \cdots.$$

As E is τ-injective, $Ext_R^1(R/I, E) = 0$. Also, since I is τ-projective and K is τ-torsion,

$$Ext_R^2(R/I, K) \simeq Ext_R^1(I, K) = 0.$$

Hence $Ext_R^1(R/I, E/K) = 0$ for all $I \in \mathscr{L}(\tau)$ and so E/K is in $\mathscr{I}\mathscr{T}_\tau$.

Conversely, let τ be a stable torsion theory. Suppose that any R-module has a monic $\mathscr{I}\mathscr{T}_\tau$-cover. By [6, Proposition 4], $\mathscr{I}\mathscr{T}_\tau$ is closed under direct sums and epimorphic images. Take X a τ-torsion R-module and $I \in \mathscr{L}(\tau)$. Applying $Hom_R(R/I, -)$ to the exact sequence $0 \to X \to E(X) \to E(X)/X \to 0$, we have:

$$\cdots \to Ext_R^1(R/I, E(X)/X) \to Ext_R^2(R/I, X) \to 0 = Ext_R^2(R/I, E(X)).$$

Since τ is stable, it follows that $E(X)$ is τ-torsion. Therefore $E(X)/X$ is τ-injective and so $Ext_R^1(R/I, E(X)/X) = 0$. We conclude that

$$0 = Ext_R^2(R/I, X) \simeq Ext_R^1(I, X),$$

and I is τ-projective.

On the other hand, since $\mathscr{I}\mathscr{T}_\tau$ is closed under direct sums, by [9, Lemma 2], $\mathscr{L}(\tau)$ verifies the ascending chain condition over ideals. □

We will say that an R-module M is $\mathscr{I}\mathscr{T}_\tau$-cotorsion (resp. $\mathscr{I}\mathscr{T}_\tau$-reduced) if $Ext_R^1(E, M) = 0$ (resp. $Hom_R(E, M) = 0$) for all $E \in \mathscr{I}\mathscr{T}_\tau$. The following result is a generalization of the well-known descomposition theorem of abelian groups in divisible and reduced part.

PROPOSITION 2. *Let τ be a torsion theory in R-Mod such that $\mathscr{L}(\tau)$ verifies the ascending chain condition over ideals.*

The following assertions are equivalent.

(a) *$\mathscr{I}\mathscr{T}_\tau$ is closed under epimorphic images.*

(b) *Every $\mathscr{I}\mathscr{T}_\tau$-cover is a monomorphism.*

(c) *$\mathscr{I}\mathscr{T}_\tau$ coincides with the class of τ-torsion and $\mathscr{I}\mathscr{T}_\tau$-cotorsion R-modules.*

(d) *Any τ-torsion R-module T is a direct sum $T = E \oplus K$ of an R-module E in $\mathscr{I}\mathscr{T}_\tau$ and a $\mathscr{I}\mathscr{T}_\tau$-reduced R-module K.*

PROOF. The equivalence of (a) and (b) is a direct consequence of [6, Proposition 4].

(a)$\Rightarrow$(c) Denote by $\mathscr{K}_\tau$ the class of τ-torsion and $\mathscr{I}\mathscr{T}_\tau$-reduced R-modules. Then, it is clear that $\mathscr{I}\mathscr{T}_\tau \subseteq \mathscr{K}_\tau$. Conversely, let $C \in \mathscr{K}_\tau$ and consider the short exact sequence $0 \to C \to E_\tau(C) \to E_\tau(C)/C \to 0$. By hypothesis, $E_\tau(C)/C \in \mathscr{I}\mathscr{T}_\tau$, then the sequence splits and so $C \in \mathscr{I}\mathscr{T}_\tau$.

(c)$\Rightarrow$(b) Let $M \in R$-Mod. We consider the exact sequence $0 \to C \to E \xrightarrow{\phi} M$, where $\phi : E \to M$ is the $\mathscr{I}\mathscr{T}_\tau$-cover of M. Then C is τ-torsion and $\mathscr{I}\mathscr{T}_\tau$-cotorsion. Therefore, by hypothesis, C is in $\mathscr{I}\mathscr{T}_\tau$. Hence the above exact sequence splits, a contradiction if $C \neq 0$. So ϕ is a monomorphism.

(c)$\Rightarrow$(d) Let T be any τ-torsion R-module. We consider the exact sequence $0 \to E \to T \to K \to 0$, where $E \to T$ is the $\mathscr{I}\mathscr{T}_\tau$-cover of T. Since K is τ-torsion the sequence splits and so T is a direct sum $T = E \oplus K$, where E is the maximal submodule of T that belongs to $\mathscr{I}\mathscr{T}_\tau$. It is clear that K is $\mathscr{I}\mathscr{T}_\tau$-reduced.

(d)$\Rightarrow$(b) If $E \to M$ is an $\mathscr{I}\mathscr{T}_\tau$-cover of M, then $E \to \tau(M)$ is an $\mathscr{I}\mathscr{T}_\tau$-cover of $\tau(M)$. By (c)$\tau(M) = E' \oplus K$ with $E' \in \mathscr{I}\mathscr{T}_\tau$ and $K \in \mathscr{K}_\tau$. Then, $E = E'$ is the cover and it is monic. $\square$

4. Homology relative to $\mathscr{I}\mathscr{T}_\tau$

In this Section we introduce left derived functors of $Hom(-,-)$ relative to the class $\mathscr{I}\mathscr{T}_\tau$.

Let τ be a torsion theory in R-Mod, and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. If $\cdots \xrightarrow{\delta_2} E_1 \xrightarrow{\delta_1} E_0 \xrightarrow{\varepsilon} N \to 0$ is a $\mathscr{I}\mathscr{T}_\tau$-resolvent of N, it is clear that the sequence

$$\cdots \xrightarrow{\delta_{2*}} Hom(E, E_1) \xrightarrow{\delta_{1*}} Hom(E, E_0) \xrightarrow{\varepsilon_*} Hom(E, N) \to 0$$

is exact for each E in $\mathscr{I}\mathscr{T}_\tau$. Also, if E is any R-module, the above sequence verifies $\delta_{k_*}\delta_{k+1_*} = 0$ for any positive integer k. It can be proved that it is possible to construct, like with projective resolvents, left derived functors of $Hom(M,-)$ relative to $\mathscr{I}\mathscr{T}_\tau$, for each R-module M. For it, we define $\tau - Ext_0(M,N) = Hom(M,E_0)/Im\delta_{1_*}$, $\tau - Ext_1(M,N) = Ker\varepsilon_*/Im\delta_{1_*}\tau - Ext_i(M,N) = Ker\delta_{i-1_*}/Im\delta_{i_*}$ for all $i > 1$. In general, $\tau - Ext_0(M,N)$ is not isomorphic to $Hom(M,N)$, as it is the case by using projective resolvents.

In order to construct $\mathscr{I}\mathscr{T}_\tau$-resolutions, we must have $\mathscr{I}\mathscr{T}_\tau$-envelopes for any R-module. So, let τ be a TTF-theory (see Theorem 2). For each R-module M, it is possible to construct a $\mathscr{I}\mathscr{T}_\tau$-resolution

$$0 \to M \xrightarrow{\eta} E^0 \xrightarrow{\delta^1} E^1 \xrightarrow{\delta^2} \dots,$$

where $C^0 = M$, and $C^i = Im\delta^i (i \geq 1)$ are the cosyzygies of the above resolution. Since E^i is the τ-injective envelope of C^i, note that, for $i \geq 1$, each C^i is a submodule of E^i.

In the same way that with $\mathscr{I}\mathscr{T}_\tau$-resolvents, now we can construct left derived functors of $Hom_R(-,N)$ by using $\mathscr{I}\mathscr{T}_\tau$-resolutions of M. We give a description of the left derived functors $\tau - \overline{Ext_i(-,N)}$ of $Hom_R(-,N)$ relative to $\mathscr{I}\mathscr{T}_\tau$. Given two R-module M and N we consider a $\mathscr{I}\mathscr{T}_\tau$-resolution of M

$$0 \to M \xrightarrow{\eta} E^0 \xrightarrow{\delta^1} E^1 \xrightarrow{\delta^2} \dots,$$

by applying $Hom(-,N)$ we obtain the complex

$$\dots \xrightarrow{\delta^{2^*}} Hom(E^1,N) \xrightarrow{\delta^{1^*}} Hom(E^0,N) \xrightarrow{\eta^*} Hom(M,N).$$

Then $\tau - \overline{Ext_0(M,N)} = Hom(E^0,N)/Im\delta^{1^*}$, $\tau - \overline{Ext_1(M,N)} = Ker\eta^*/Im\delta^{1^*}$ and $\tau - \overline{Ext_i(M,N)} = Ker\delta^{i-1^*}/Im\delta^{i^*}$ for $i > 1$.

We need to show that left derived functors given above are well defined and that $\tau - Ext_i(M,N) = \tau - \overline{Ext_i(M,N)}$ for all i.

Let $\mathscr{A}$ and $\mathscr{B}$ abelian categories and $\mathscr{C}$, $\mathscr{D}$ full subcategories of $\mathscr{A}$ and $\mathscr{B}$ respectively. Following [5], we will say that the additive functor $F: \mathscr{A} \times \mathscr{B} \to Ab$ (contravariant in the first variable and covariant in the second one) is left balanced relative to $(\mathscr{C},\mathscr{D})$ if for each object A of $\mathscr{A}$ there is a complex

$$0 \to A \to C^0 \to C^1 \to \dots,$$

with each $C^i \in \mathscr{C}$ which becomes exact when $F(-,D)$ is applied for any $D \in \mathscr{D}$, and if for each object B of $\mathscr{B}$ there is a complex

$$\dots \to D_1 \to D_0 \to B \to 0$$

with each D_i in $\mathscr{D}$ such that the functor $F(C,-)$ applied to the complex gives an exact sequence whenever $C \in \mathscr{C}$.

PROPOSITION 3. a) *Let τ be a torsion theory, and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. Then*

$$Hom(-,-)\colon \mathscr{T}_\tau \times R-Mod \to Ab$$

is a left balanced functor by $(\mathscr{I}\mathscr{T}_\tau,\ \mathscr{I}\mathscr{T}_\tau)$.

b) *Let τ be a TTF-theory, and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. Then*

$$Hom(-,-)\colon R-Mod \times R-Mod \to Ab$$

is a left balanced functor by $(\mathscr{I}\mathscr{T}_\tau,\ \mathscr{I}\mathscr{T}_\tau)$.

PROOF. Apply Theorem 1 and Theorem 2. □

Then, if τ is a TTF-theory and $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals, M and N are two R-modules such that M has a $\mathscr{I}\mathscr{T}_\tau$-resolution $\{E^i,\delta^i\}$ and N has a $\mathscr{I}\mathscr{T}_\tau$-resolvent $\{E_i,\delta_i\}$, then the double complex $Hom(E^n,E_m)$, and the complexes $Hom(E^n,N)$ and $Hom(M,E_n)$ have isomorphic homology, [5, Proposition 2.3]. We will denote by $\tau - Ext_n(M,N)$ the homology of the complex $Hom(M,E_n)$ (or the homology of the complex $Hom(E^n,N)$).

By [5, Corollary 2.4], the definitions of the relative homology functors given before do not depend on the $\mathscr{I}\mathscr{T}_\tau$-resolvents and $\mathscr{I}\mathscr{T}_\tau$-resolutions taken.

Following [5], with some modifications, we give the following definitions.

DEFINITION 3. a) *Let M be an R-module and $\mathscr{C}$ a full subcategory of R-Mod. We define $\mathscr{C} - l.dim(M)$, the dimension respect to $\mathscr{C}$-resolvents of M, as the less positive integer n such that there exist a $\mathscr{C}$-resolvent*

$$0 \to E_{n-1} \to E_{n-2} \to \cdots \to E_0 \to M \to 0,$$

if such integer exists. We say that $\mathscr{C} - l.dim(M) < \infty$ if $\mathscr{C} - l.dim(M) = n$ for some non negative integer n.

Dually, the dimension respect to $\mathscr{C}$-resolutions of M is defined. It will be denoted by $\mathscr{C} - r.dim(M)$.

b) *The left global dimension of R relative to $\mathscr{C}$ is defined as the supremum of $\mathscr{C} - l.dim(M)$, for all R-modules M. It is denoted by $\mathscr{C}^l$-$gl.dimR$.*

Dually the right global dimension of R relative to $\mathscr{C}$ is defined as the supremum of $\mathscr{C} - r.dim\ (M)$, for all R-modules M. It is denoted by $\mathscr{C}^r - gl.dimR$.

Lemma 4. a) *Let τ be a torsion theory, and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals.*

a.1) *The following conditions are equivalent.*

i) $\mathscr{I}\mathscr{T}_\tau - l.dim(M) = 0$.

ii) *M is $\mathscr{I}\mathscr{T}_\tau$-reduced.*

iii) $\tau - Ext_0(X, M) = 0$ *for all* $X \in R$-Mod.

a.2) *The following assertions are equivalent.*

i) $\mathscr{I}\mathscr{T}_\tau - l.dim(M) = 1$.

ii) *M is not $\mathscr{I}\mathscr{T}_\tau$-reduced and there exists a $\mathscr{I}\mathscr{T}_\tau$-precover of M, $\varepsilon: E \to M$, with $Ker \in \mathscr{I}\mathscr{T}_\tau$-reduced.*

iii) $\tau - Ext_0(N, M) \neq 0$ *for some* $N \in R$-*Mod, and* $\tau - Ext_1(X, M) = 0$ *for all* $X \in R$-*Mod.*

b) *Let $\tau \equiv (\mathscr{C}, \mathscr{T}, \mathscr{F})$ be a TTF-theory in R-Mod.*

b.1) *The following conditions are equivalent.*

i) $\mathscr{I}\mathscr{T}_\tau - r.dim(M) = 0$

ii) $M \in \mathscr{C}$.

iii) $\tau - \overline{Ext_0(N, X)} = 0$ *for all* $X \in R$-*Mod.*

b.2) *Suppose, in addition that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. The following conditions are equivalent.*

i) $M \in \mathscr{I}\mathscr{T}_\tau$

ii) $\tau - Ext_0(M, X) = Hom_R(M, X)$ *for all* $X \in R$-*Mod.*

iii) $\tau - Ext_0(X, M) = Hom_R(X, M)$ *for all* $X \in R$-*Mod.*

b.3) *The following assertions are equivalent.*

i) $\mathscr{I}\mathscr{T}_\tau - r.dim(M) = 1$.

ii) *There exists a epic $\mathscr{I}\mathscr{T}_\tau$-preenvelope of M.*

iii) $\tau - \overline{Ext_0(M, N)} \neq 0$ *for some* $N \in R$-*Mod, and* $\tau - \overline{Ext_1(M, X)} = 0$ *for all* $X \in R$-*Mod.*

Proof. We will prove (a). (b) can be proved with similar arguments.

(a.1) i) $\Rightarrow$ ii) If $\mathscr{I}\mathscr{T}_\tau - l.dim(M) = 0$, then any $\mathscr{I}\mathscr{T}_\tau$-precover of M is zero. So, it is clear that $Hom(E, M) = 0$ for each $E \in \mathscr{I}\mathscr{T}_\tau$.

ii) $\Rightarrow$ iii) By (ii) $0 \to M$ is a $\mathscr{I}\mathscr{T}_\tau$-resolvent of M. Therefore, we have $\tau - Ext_0(X, M) = Hom(X, 0) = 0$, for all $X \in R$-Mod.

iii) $\Rightarrow$ i) It is enough to check that $0 \to M$ is a $\mathscr{I}\mathscr{T}_\tau$-precover of M. Let $\cdots \xrightarrow{\delta_2} E_1 \xrightarrow{\delta_1} E_0 \xrightarrow{\varepsilon} M \to 0$ be a $\mathscr{I}\mathscr{T}_\tau$-resolvent of M. We take any R-module. Then, appliying $Hom(X, -)$ to the above complex, we obtain the complex

$$\cdots \xrightarrow{\delta_{2*}} Hom(X, E_1) \xrightarrow{\delta_{1*}} Hom(X, E_0) \xrightarrow{\varepsilon_*} Hom(X, M) \to 0.$$

Therefore, $\tau - Ext_0(X, M) = Hom(X, E_0)/Im\delta_{1_*} = 0$. Hence, for $E \in \mathscr{I}\mathscr{T}_\tau$ we have the exact sequence

$$0 \to Hom(E, C_1) \to Hom(E, E_1) \xrightarrow{\delta_{1_*}} Hom(E, C_0) \to 0,$$

where C_i are the syzygies of the above $\mathscr{I}\mathscr{T}_\tau$-resolvent. Then $Hom(E, C_0) = Im\delta_{1_*}$. It implies that

$$Hom(E, M) = Hom(E, E_0)/Hom(E, C_0) = \tau - Ext_0(E, M) = 0.$$

So, M is $\mathscr{I}\mathscr{T}_\tau$-reduced.

(a.2) i)$\Rightarrow$ii) (i) implies that there exists a $\mathscr{I}\mathscr{T}_\tau$-resolvent of M in the form $0 \to E_0 \to M$, with $E_0 \neq 0$. Then M is not $\mathscr{I}\mathscr{T}_\tau$-reduced. If the kernel of any $\mathscr{I}\mathscr{T}_\tau$-precover is not $\mathscr{I}\mathscr{T}_\tau$-reduced, then any $\mathscr{I}\mathscr{T}_\tau$-resolvent of M has the form $\cdots E_1 \to E_0 \to M \to 0$ with E_0, $E_1 \neq 0$. This is a contradiction with (i).

ii)$\Rightarrow$iii) If M is not $\mathscr{I}\mathscr{T}_\tau$-reduced, then, by (a.1), $\tau - Ext_0(N, M) \neq 0$ for some $N \in R$-Mod. Now, we will prove that $\tau - Ext_1(X, M) = 0$, for all $X \in R$-Mod. We take a $\mathscr{I}\mathscr{T}_\tau$-precover of M which kernel is $\mathscr{I}\mathscr{T}_\tau$-reduced, $\varepsilon: E_0 \to M$. Then $0 \xrightarrow{\delta_1} E_0 \to M$ is a $\mathscr{I}\mathscr{T}_\tau$-resolvent of M. So, $\tau - Ext_1(X, M) = Hom(X, 0)/Im\delta_{1_*} = 0$, for all $X \in R$-Mod.

iii)$\Rightarrow$i) We show that there exists a $\mathscr{I}\mathscr{T}_\tau$-resolvent of M in the form $0 \xrightarrow{\delta_1} E_0 \xrightarrow{\varepsilon} M \to 0$. We consider a minimal $\mathscr{I}\mathscr{T}_\tau$-resolvent of M:

$$\cdots \xrightarrow{\delta_2} E_1 \xrightarrow{\delta_1} E_0 \xrightarrow{\varepsilon} M \to 0.$$

For any $X \in R$-Mod, we have the exact sequence $0 \to Hom(X, C_1) \to Hom(X, E_1) \xrightarrow{\delta_{1_*}} Hom(X, C_0) \to 0(\tau - Ext_1(X, M) = Ker\varepsilon_*/Im\delta_{1_*} = 0)$. Therefore, the sequence $0 \to C_1 \to E_1 \to C_0 \to 0$ is splitting. Hence $C_0 \in \mathscr{I}\mathscr{T}_\tau$ and so $C_0 = 0$ (see [6, Proposition 2]). So $0 \to E_0 \to M$ is a $\mathscr{I}\mathscr{T}_\tau$-resolvent of M. □

Now, by means of an inductive argument we can deduce the following result.

THEOREM 3. *Let τ be a torsion theory, and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. Let M be an R-module. The following conditions are equivalent for $n \geq 2$.*

(a) *$\mathscr{I}\mathscr{T}_\tau - l.dim(M) = n$.*

(b) *There exists a $\mathscr{I}\mathscr{T}_\tau$-resolvent of M such that the ith syzygy is not $\mathscr{I}\mathscr{T}_\tau$-reduced for $i \leq n-2$ and the $n-1$th syzygy is $\mathscr{I}\mathscr{T}_\tau$-reduced.*

c) *$\tau - Ext_i(N, M) \neq 0$ for all $i < n$ and some $N \in R$-Mod and $\tau - Ext_n(X, M) = 0$ for any R-module X.*

Let τ be a TTF-theory in R-Mod. Dually, the following conditions are equivalent for $n \geq 2$.

(a) $\mathscr{I}\mathscr{T}_\tau - r.dim(M) = n$.

(b) *There exists a $\mathscr{I}\mathscr{T}_\tau$-resolution of M such that the ith cosyzygy do not belong to $\mathscr{I}\mathscr{T}_\tau$ for $i \leq n-2$ and the $n-1$th cosyzygy belongs to $\mathscr{I}\mathscr{T}_\tau$.*

(c) $\tau - \overline{Ext_i(M,N)} \neq 0$ *for all $i > n$ and some $N \in$ R-Mod, and* $\tau - \overline{Ext_n(M,X)} = 0$, *for each R-module X.*

Remark. If the syzygy C_i is $\mathscr{I}\mathscr{T}_\tau$-reduced, then the $\mathscr{I}\mathscr{T}_\tau$-precover $E_i \rightarrow C_{i-1}$ is a $\mathscr{I}\mathscr{T}_\tau$-cover.

The following result is consequence of Lemma 4.

Lemma 5. *Let τ be a TTF-theory and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. The following assertions are equivalent.*

a) $\mathscr{I}\mathscr{T}_\tau^r - gl.dim(R) = 0$.

b) $\mathscr{I}\mathscr{T}_\tau^l - gl.dim(R) = 0$.

c) $\mathscr{T}_\tau = \{0\}$.

Proposition 4. *Let τ be a TTF-theory and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. Then,*

$$\mathscr{I}\mathscr{T}_\tau^r - gl.dim(R) = 2 + \mathrm{IT}_\tau^r - gl.dim(R),$$

if $\mathscr{I}\mathscr{T}_\tau^r - gl.dim(R) \geq 2$.

Proof. In order to calculate $\mathscr{I}\mathscr{T}_\tau^r - gl.dim(R)$ when τ is a non trivial TTF-theory given by the idempotent two-sided ideal I, it is convenient to remark that the localizing subcategory of τ-torsion R-modules, $\mathscr{T}_\tau$, is equivalent to the category of R/I-modules R/I-Mod. Also injectives objects in $\mathscr{T}_\tau$ (and so in R/I-Mod) are precisely the τ-torsion τ-injectives R-modules. Therefore we have

$$\mathscr{I}\mathscr{T}_\tau^r - gl.dim(R) = 1 + l.gl.dim(R/I),$$

where $l.gl.dim(R/I)$ denotes the usual left usual global dimension of R/I. On the other hand, since the $\mathscr{I}\mathscr{T}_\tau$-cover of a left R-module M is the same that the $\mathscr{I}\mathscr{T}_\tau$-cover of $\tau(M)$, we have, for τ non trivial,

$$\mathscr{I}\mathscr{T}_\tau^l - gl.dim(R) = \mathscr{I}_{R/I}^l - gl.dim(R/I),$$

where $\mathcal{I}_{R/I}$ denotes the class of injectives R/I-modules. By using [3, Proposition 8.1],

$$\mathcal{I}\mathcal{T}_\tau^l - gl.dim(R) = l.gl.dim(R/I) - 1,$$

if $l.gl.dim(R/I) \geq 3$, and $\mathcal{I}\mathcal{T}_\tau^l - gl.dim(R) = 1$ if the usual left global dimension of R/I is zero, one or two. Therefore,

$$\mathcal{I}\mathcal{T}_\tau^r - gl.dim(R) = 2 + \mathcal{I}\mathcal{T}_\tau^l - gl.dim(R),$$

if $\mathcal{I}\mathcal{T}_\tau^l - gl.dim(R) \geq 2$. □

When τ is an arbitrary hereditary torsion theory, the dimension $\mathcal{I}\mathcal{T}_\tau^l - gl.dim(R)$ coincides with the supremum of the length of injective resolvents in the full subcategory $\mathcal{T}_\tau$ of R-Mod. The following result gives sufficient and necessary conditions for the case $\mathcal{I}\mathcal{T}_\tau^l - gl.dim(R) = 1$.

In [5, Pag. 307], Enochs and Jenda have characterized the coreflexivity of the full subcategory of injectives R-modules of R-Mod, for R any left noetherian ring, in terms of the usual left global dimension of R. The following Proposition extends the above result. It is proved that the full subcategory $\mathcal{I}\mathcal{T}_\tau$ of injective objects in $\mathcal{T}_\tau$ is a Co-Giraud subcategory of $\mathcal{T}_\tau$. It means that $\mathcal{I}\mathcal{T}_\tau$ is a coreflexive subcategory with a preserving co-kernel coreflector $\mathscr{C}^\tau$ (see [11]).

Proposition 5. *Let τ be a torsion theory and suppose that $\mathscr{L}(\tau)$ verifies the ascending chain condition over left ideals. The following conditions are equivalent.*

(a) $\mathcal{I}\mathcal{T}_\tau^l - gl.dim(R) \leq 1$.

(b) *The inclusion functor, $i : \mathcal{I}\mathcal{T}_\tau \to \mathcal{T}_\tau$, has the right adjoint $\mathscr{C}^\tau : \mathcal{T}_\tau \to \mathcal{I}\mathcal{T}_\tau$, where $\mathscr{C}^\tau(M)$ is the $\mathcal{I}\mathcal{T}_\tau$-cover of M for all $M \in \mathcal{T}_\tau$.*

Proof. (a) $\Rightarrow$ (b) By Theorem 3, any kernel of a $\mathcal{I}\mathcal{T}_\tau$-precover is $\mathcal{I}\mathcal{T}_\tau$-reduced. Given $T \in \mathcal{T}_\tau$, we consider the exact sequence $0 \to K \to \mathscr{C}^\tau(T) \to T$, where $\mathscr{C}^\tau(T) \to T$ is the $\mathcal{I}\mathcal{T}_\tau$-cover of T. Then, for all $E \in \mathcal{I}\mathcal{T}_\tau$, by applying $Hom_R(E, -)$, we obtain the natural isomorphism $Hom_R(i(E), T) \cong Hom_R(E, \mathscr{C}^\tau(T))$. So $\mathscr{C}^\tau$ is right adjoint of i.

(b) $\Rightarrow$ (a) By Theorem 3, it is enough to check that any non trivial $\mathcal{I}\mathcal{T}_\tau$-cover has a $\mathcal{I}\mathcal{T}_\tau$-reduced kernel. But, this can be proved by the reverse argument of the above. □

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